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Uniqueness of Gaussian quadrature formula for computed tomography[☆]

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Abstract

This note answers positively a question raised by B. Bojanov and G. Petrova. Namely, the Gaussian quadrature formula for computed tomography among the given type is unique.

Keywords: Radon transforms; Gaussian quadrature; Tchebycheff polynomials of second kind

1. Introduction

The Radon transform $\mathcal{R}(f; t, \theta), t \in (-1, 1), \theta \in [0, \pi)$ of a function f, defined on the unit ball $\mathbf{B} := \{(x, y) : x^2 + y^2 \le 1\}$, is given by the integral of f along the line segment $I(t, \theta) := \{(x, y) : x \cos\theta + y \sin\theta = t\} \bigcap \mathbf{B}$, namely,

$$\mathcal{R}(f; t, \theta) := \int_{I(t,\theta)} f(x, y)\mathrm{d}s$$
$$= \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\mathrm{d}s.$$

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We consider the space $L_2(\mathbf{B})$ of bivariate square integrable functions defined on $\mathbf{B}$. It is a well known fact (see [1]) that the set of polynomials $\{U_{k,n}\}_{n=0,}^{\infty}{}_{k=0}^{n}$, defined by

$$U_{k,n}(x, y) := \frac{1}{\sqrt{\pi}} U_n(x \cos(\theta_{k,n}) + y \sin(\theta_{k,n})), \quad \theta_{k,n} := \frac{k\pi}{n+1},$$

where $U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$ is the Tchebycheff polynomial of second kind, form a complete orthonormal system for $L_2(\mathbf{B})$. It can be shown that the coefficients $c_{k,n}(f)$ in the expansion of f,

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_{k,n}(f) U_n(x \cos(\theta_{k,n}) + y \sin(\theta_{k,n})),$$

with respect to this system are

$$c_{k,n}(f) := \frac{1}{\sqrt{\pi}} \int_{\mathbf{B}} f(x, y) U_{k,n}(x, y) \mathrm{d}x \mathrm{d}y = \frac{1}{\pi} \int_{-1}^{1} \mathcal{R}(f; t, \theta_{k,n}) U_n(t) \mathrm{d}t. \tag{1}$$

In view of formula (1), the problem of optimal recovery of f is equivalent to the problem of a selection of quadrature for the second integral in (1) which is exact for polynomials of degree as high as possible. In [2], quadrature formulas of type

$$\int_{\mathbf{B}} f(x, y) U_n(x \cos\theta + y \sin\theta) \mathrm{d}x \mathrm{d}y \approx \sum_{j=1}^{n+1} b_j \mathcal{R}(f; \xi_j, \theta), \tag{2}$$

with nodes ξ_j and coefficients b_j, are considered. It is clear that the algebraic degree of precision (ADP) of (2) is no more than $3n + 1$. Formulas of type (2) with ADP $= 3n + 1$ are called Gaussian. It is shown in [2] that the following formula is Gaussian,

$$\int_{\mathbf{B}} f(x, y) U_n(x \cos\theta + y \sin\theta) \mathrm{d}x \mathrm{d}y$$
$$\approx \frac{\pi}{2n+2} \sum_{j=1}^{n+1} (-1)^{j-1} \mathcal{R}\left(f; \cos\frac{(2j-1)\pi}{2n+2}, \theta\right). \tag{3}$$

It is also questioned in [2] whether formula (3) is the only Gaussian formula among formulas of type

$$\int_{\mathbf{B}} f(x, y) U_n(x \cos\theta + y \sin\theta) \mathrm{d}x \mathrm{d}y \approx \sum_{j=1}^{n+1} b_j \mathcal{R}(f; \xi_j, \theta_j), \tag{4}$$

where the Radon transforms are possibly taken not along parallel lines.

In this note, we will answer it positively. Namely, formula (3) is the unique quadrature formula with ADP $= 3n + 1$ among formulas of type (4).

2. Proof of the uniqueness

We first prove a useful lemma.

Lemma. *For given $0 \le \theta_0 < \theta_1 < \cdots < \theta_m < \pi$, $\{U_s(\cos(\theta - \theta_j))\}_{j=0}^{m}$ are linearly independent provided that $s \ge m$.*

Proof. Assume that $\sum_{j=0}^{m} b_j U_s(\cos(\theta - \theta_j)) \equiv 0$. We only need to show that $b_j = 0$, $j = 0, 1, \ldots, m$. Noting that

$$U_{2k}(\cos\theta) = 1 + 2\sum_{j=1}^{k} \cos 2j\theta, \qquad U_{2k-1}(\cos\theta) = 2\sum_{j=1}^{k} \cos(2j-1)\theta, \tag{5}$$

we have for $s = 2k$,

$$\sum_{j=0}^{m} b_j + \sum_{l=1}^{k} \cos 2l\theta \sum_{j=0}^{m} 2b_j \cos 2l\theta_j + \sum_{l=1}^{k} \sin 2l\theta \sum_{j=0}^{m} 2b_j \sin 2l\theta_j \equiv 0.$$

Therefore,

$$\begin{cases} \sum_{j=0}^{m} b_j = 0 \\ \sum_{j=0}^{m} b_j \cos 2l\theta_j = 0 \\ \sum_{j=0}^{m} b_j \sin 2l\theta_j = 0, \quad l = 1, 2, \ldots, k. \end{cases}$$

For completing the proof for the case $s = 2k$, we only need to show that rank $A = m + 1$, where

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \cos 2\theta_0 & \cos 2\theta_1 & \cos 2\theta_2 & \cdots & \cos 2\theta_m \\ \sin 2\theta_0 & \sin 2\theta_1 & \sin 2\theta_2 & \cdots & \sin 2\theta_m \\ \vdots & \vdots & \vdots & & \vdots \\ \cos 2k\theta_0 & \cos 2k\theta_1 & \cos 2k\theta_2 & \cdots & \cos 2k\theta_m \\ \sin 2k\theta_0 & \sin 2k\theta_1 & \sin 2k\theta_2 & \cdots & \sin 2k\theta_m \end{pmatrix}.$$

For that, we apply on A a series of elementary row and column operations as follows.

Step 1: for $p = 1, \ldots, k$, multiply the $(2p+1)$th row by i $\left(\mathrm{i} = \sqrt{-1}\right)$ and add it to the $2p$th row, then multiply the $(2p+1)$th row by $-2\mathrm{i}$ and add the $2p$th row to it. We obtain the matrix A', where

$$A' = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \mathrm{e}^{2\theta_0 \mathrm{i}} & \mathrm{e}^{2\theta_1 \mathrm{i}} & \mathrm{e}^{2\theta_2 \mathrm{i}} & \cdots & \mathrm{e}^{2\theta_m \mathrm{i}} \\ \mathrm{e}^{-2\theta_0 \mathrm{i}} & \mathrm{e}^{-2\theta_1 \mathrm{i}} & \mathrm{e}^{-2\theta_2 \mathrm{i}} & \cdots & \mathrm{e}^{-2\theta_m \mathrm{i}} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathrm{e}^{2k\theta_0 \mathrm{i}} & \mathrm{e}^{2k\theta_1 \mathrm{i}} & \mathrm{e}^{2k\theta_2 \mathrm{i}} & \cdots & \mathrm{e}^{2k\theta_m \mathrm{i}} \\ \mathrm{e}^{-2k\theta_0 \mathrm{i}} & \mathrm{e}^{-2k\theta_1 \mathrm{i}} & \mathrm{e}^{-2k\theta_2 \mathrm{i}} & \cdots & \mathrm{e}^{-2k\theta_m \mathrm{i}} \end{pmatrix}.$$

Step 2: for $q = 1, \ldots, m+1$, multiply the qth column of A' by $\mathrm{e}^{2k\theta_{q-1}\mathrm{i}}$, obtaining the new matrix A''.

Step 3: interchange the rows of A'' properly, deriving

$$A''' = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ h_0 & h_1 & h_2 & \cdots & h_m \\ h_0^2 & h_1^2 & h_2^2 & \cdots & h_m^2 \\ \vdots & \vdots & \vdots & & \vdots \\ h_0^{2k} & h_1^{2k} & h_2^{2k} & \cdots & h_m^{2k} \end{pmatrix}, \quad h_j = \mathrm{e}^{2\theta_j \mathrm{i}},\ j = 0, 1, \ldots, m,$$

which has the same rank as A. Since $\theta_j \in [0, \pi)$ $(j = 0, 1, \ldots, m)$, $h_j \neq h_l$ for any $j, l \in \{0, 1, \ldots, m\}$ and $j \neq l$. Therefore, the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ h_0 & h_1 & h_2 & \cdots & h_m \\ h_0^2 & h_1^2 & h_2^2 & \cdots & h_m^2 \\ \vdots & \vdots & \vdots & & \vdots \\ h_0^{m} & h_1^{m} & h_2^{m} & \cdots & h_m^{m} \end{pmatrix},$$

as a sub-matrix of A''' for $2k \geq m$, is non-zero, which means that rank A = rank $A''' = m + 1$.

For $s = 2k - 1$, we have by the assumption and (5),

$$\sum_{l=1}^{k} \cos(2l-1)\theta \sum_{j=0}^{m} 2b_j \cos(2l-1)\theta_j + \sum_{l=1}^{k} \sin(2l-1)\theta \sum_{j=0}^{m} 2b_j \sin(2l-1)\theta_j \equiv 0.$$

Therefore,

$$\begin{cases} \displaystyle\sum_{j=0}^{m} b_j \cos(2l-1)\theta_j = 0 \\ \displaystyle\sum_{j=0}^{m} b_j \sin(2l-1)\theta_j = 0, \quad l = 1, 2, \ldots, k. \end{cases}$$

We only need to show that rank $C = m + 1$, where

$$C = \begin{pmatrix} \cos\theta_0 & \cos\theta_1 & \cos\theta_2 & \cdots & \cos\theta_m \\ \sin\theta_0 & \sin\theta_1 & \sin\theta_2 & \cdots & \sin\theta_m \\ \cos 3\theta_0 & \cos 3\theta_1 & \cos 3\theta_2 & \cdots & \cos 3\theta_m \\ \sin 3\theta_0 & \sin 3\theta_1 & \sin 3\theta_2 & \cdots & \sin 3\theta_m \\ \vdots & \vdots & \vdots & & \vdots \\ \cos(2k-1)\theta_0 & \cos(2k-1)\theta_1 & \cos(2k-1)\theta_2 & \cdots & \cos(2k-1)\theta_m \\ \sin(2k-1)\theta_0 & \sin(2k-1)\theta_1 & \sin(2k-1)\theta_2 & \cdots & \sin(2k-1)\theta_m \end{pmatrix}.$$

For that, we apply on C a series of elementary row and column operations as follows.

Step 1: for $p = 1, \ldots, k$, multiply the $2p$th row by i $\left(\mathrm{i} = \sqrt{-1}\right)$ and add it to the $(2p-1)$th row, then multiply the $2p$th row by $-2\mathrm{i}$ and add the $(2p-1)$th row to it, obtaining the new matrix C', where

$$C' = \begin{pmatrix} e^{\theta_0 i} & e^{\theta_1 i} & e^{\theta_2 i} & \cdots & e^{\theta_m i} \\ e^{-\theta_0 i} & e^{-\theta_1 i} & e^{-\theta_2 i} & \cdots & e^{-\theta_m i} \\ e^{3\theta_0 i} & e^{3\theta_1 i} & e^{3\theta_2 i} & \cdots & e^{3\theta_m i} \\ e^{-3\theta_0 i} & e^{-3\theta_1 i} & e^{-3\theta_2 i} & \cdots & e^{-3\theta_m i} \\ \vdots & \vdots & \vdots & & \vdots \\ e^{(2k-1)\theta_0 i} & e^{(2k-1)\theta_1 i} & e^{(2k-1)\theta_2 i} & \cdots & e^{(2k-1)\theta_m i} \\ e^{-(2k-1)\theta_0 i} & e^{-(2k-1)\theta_1 i} & e^{-(2k-1)\theta_2 i} & \cdots & e^{-(2k-1)\theta_m i} \end{pmatrix}.$$

Step 2: for $q = 1, \dots, m+1$, multiply the qth column of C' by $e^{(2k-1)\theta_{q-1} i}$, deriving the matrix C''.

Step 3: interchange the rows of C'' properly, obtaining

$$C''' = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ h_0 & h_1 & h_2 & \cdots & h_m \\ h_0^2 & h_1^2 & h_2^2 & \cdots & h_m^2 \\ \vdots & \vdots & \vdots & & \vdots \\ h_0^{2k-1} & h_1^{2k-1} & h_2^{2k-1} & \cdots & h_m^{2k-1} \end{pmatrix}, \quad h_j = e^{2\theta_j i}, \ j = 0, 1, \dots, m,$$

which has the same rank as C. We can similarly show that $\operatorname{rank} C''' = m + 1$. The proof is complete. $\square$

We now prove that quadrature formula of type (4) with ADP $= 3n + 1$ is unique. It is always formula (3). Based on the results in [2], we only need to show that each quadrature formula of type (4) with at least two different θ_j has ADP $< 3n + 1$. For this purpose, we can generally assume that a quadrature formula of type (4) is

$$\int_{\mathbf{B}} f(x, y) U_n(x\cos\theta + y\sin\theta) \mathrm{d}x\mathrm{d}y \approx \sum_{j=1}^{r} \sum_{l=1}^{n_j} b_{jl} \mathcal{R}(f; \xi_{jl}, \theta_j), \tag{6}$$

where $0 \le \theta_1 < \theta_2 < \cdots < \theta_r < \pi$, $\sum_{j=1}^{r} n_j = n + 1, 2 \le r \le n + 1$, and $\xi_{jl} \in (-1, 1), l = 1, \dots, n_j$, are different from each other for fixed j. Suppose that the ADP of formula (6) is $3n + 1$. We will obtain a contradiction by showing that some coefficients in (6) must be zero. Since (see [3])

$$\int_{\mathbf{B}} U_l(x\cos\theta + y\sin\theta) U_m(x\cos\phi + y\sin\phi) \mathrm{d}x\mathrm{d}y = 0, \quad \forall 0 \le \theta,\ \phi < \pi,\ l \ne m,$$

and

$$\begin{aligned} &\mathcal{R}(U_m(x\cos\phi + y\sin\phi); \xi, \theta) = \sqrt{1 - \xi^2} U_m(\xi) U_m(\cos(\theta - \phi)), \\ &\quad \forall m, \xi \in (-1, 1),\ 0 \le \theta,\ \phi < \pi, \end{aligned}$$

we have for all $n + 1 \le m \le 3n + 1$,

$$\begin{aligned} 0 &= \int_{\mathbf{B}} U_m(x\cos\phi + y\sin\phi) U_n(x\cos\theta + y\sin\theta) \mathrm{d}x\mathrm{d}y \\ &= \sum_{j=1}^{r} \sum_{l=1}^{n_j} b_{jl} \sqrt{1 - \xi_{jl}} U_m(\xi_{jl}) U_m(\cos(\phi - \theta_j)). \end{aligned}$$

By the lemma, since $m \geq n+1 \geq r > r-1$, we have

$$\sum_{l=1}^{n_j} b_{jl}\sqrt{1-\xi_{jl}}U_m(\xi_{jl}) = 0,$$

for all $n+1 \leq m \leq 3n+1$, and $j = 1, \ldots, r$. In the following, we will show that there always exists some b_{jl} being zero in the above equations. Since $r \geq 2$, there exists at least one n_j such that $n_j \leq \left[\frac{n}{2}\right] + 1$ for some $j \in \{1, \ldots, r\}$. Therefore, we can briefly assume that

$$\sum_{l=1}^{p} d_l U_m(\xi_l) = 0, \quad n+1 \leq m \leq 3n+1, \tag{7}$$

for some $n_j = p \leq \left[\frac{n}{2}\right] + 1$, $d_l = d_{jl} = b_{jl}\sqrt{1-\xi_{jl}}$, and $\xi_l = \xi_{jl}$. We only need to show that there always exists some d_l being zero in (7).

Noting the three-term recurrence relations of $\{U_m\}$ (see [4]),

$$U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), \quad U_{-1}(t) = 0, \ m = 0, 1, \ldots,$$

for $q = 1, 2, \ldots, 2n-1$, we add the qth equation to the $(q+2)$th equation in system (7), obtaining

$$\sum_{l=1}^{p} d_l \xi_l U_m(\xi_l) = 0, \quad n+2 \leq m \leq 3n. \tag{8}$$

Then, for $q = 1, 2, \ldots, 2n-3$, we add the qth equation to the $(q+2)$th equation in system (8), obtaining

$$\sum_{l=1}^{p} d_l \xi_l^2 U_m(\xi_l) = 0, \quad n+3 \leq m \leq 3n-1.$$

Generally, we have the following systems for $q = 0, 1, \ldots, p-1$, $p-1 \leq \left[\frac{n}{2}\right]$,

$$\sum_{l=1}^{p} d_l \xi_l^q U_m(\xi_l) = 0, \quad n+1+q \leq m \leq 3n+1-q.$$

Especially, we have for $n+p \leq m \leq 3n+2-p$,

$$\sum_{l=1}^{p} d_l \xi_l^q U_m(\xi_l) = 0, \quad q = 0, 1, \ldots, p-1, \tag{9}$$

where the coefficient matrices are

$$D_m = \begin{pmatrix} U_m(\xi_1) & U_m(\xi_2) & \cdots & U_m(\xi_p) \\ \xi_1 U_m(\xi_1) & \xi_2 U_m(\xi_2) & \cdots & \xi_p U_m(\xi_p) \\ \vdots & \vdots & & \vdots \\ \xi_1^{p-1} U_m(\xi_1) & \xi_2^{p-1} U_m(\xi_2) & \cdots & \xi_p^{p-1} U_m(\xi_p) \end{pmatrix}.$$

We consider two systems corresponding to $m = m_1 = n+p$ and $m = m_2 = m_1 + 1$. Since U_{m_1} and U_{m_2} as orthogonal polynomials have no common zeros, we have

(i) if $U_{m_1}(\xi_l) = 0, l = 1, 2, \ldots, p$, $U_{m_2}(\xi_l) \neq 0, l = 1, 2, \ldots, p$, therefore rank $D_{m_2} = p$, which leads to $d_l = 0, l = 1, 2, \ldots, p$;
(ii) if $U_{m_1}(\xi_l) \neq 0, l = 1, 2, \ldots, p$, rank $D_{m_1} = p$, which also leads to $d_l = 0, l = 1, 2, \ldots, p$;
(iii) without loss of generality, if $U_{m_1}(\xi_l) = 0$, $1 \le l \le s < p$, and $U_{m_1}(\xi_l) \neq 0$, $s < l \le p$, for some s, we consider the first $p - s$ equations of (9) for $m = m_1$,

$$\sum_{l=s+1}^{p} d_l U_{m_1}(\xi_l)\xi_l^q = 0, \quad q = 0, \ldots, p - s - 1,$$

which has the following coefficient matrix

$$D'_{m_1} = \begin{pmatrix} U_{m_1}(\xi_{s+1}) & \cdots & U_{m_1}(\xi_p) \\ \xi_{s+1} U_{m_1}(\xi_{s+1}) & \cdots & \xi_p U_{m_1}(\xi_p) \\ \vdots & & \vdots \\ \xi_{s+1}^{p-s-1} U_{m_1}(\xi_{s+1}) & \cdots & \xi_p^{p-s-1} U_{m_1}(\xi_p) \end{pmatrix},$$

with rank $D'_{m_1} = p - s$. It leads to $d_l = 0, l = s + 1, \ldots, p$. The proof is complete. □

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